Interaction Measures for Nonsquare Decentralized Control Structures

A block relative gain measure for nonsquare systems has been derived. Evaluated at steady state, this measure serves as a performance-related tool for evaluating control structures prior to controller design. Both the nonsquare dynamic block relative gain and the relative sensitivities have been presented as dynamic interaction measures that depend on the controller tuning and are consequently applicable to design of nonsquare decentralized controllers. Rigorous relationships between the dynamic block relative gain and the performance and stability of a closed-loop system have been established.

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Introduction

One of the first tasks encountered in the synthesis of a control system is the specification of the control structure. That is, once the inputs to be manipulated and the outputs to be measured have been chosen, one must then decide how they should be connected to achieve feedback. For the square systems traditionally considered, many alternatives face the designer, ranging from a fully centralized system to a completely decentralized system. The fully centralized controller would use feedback from all of the measured outputs to manipulate each input. Conversely, a fully decentralized controller would pair one output with one input and implement feedback-based control on each pair independently. While the fully centralized controller inherently utilizes more information about the system, it is much harder to implement and more vulnerable to hardware failure than a decentralized system. However, the completely decentralized system can lead to a high degree of interactions because of the information neglected by its structure. In such cases a block decentralized structure would be preferred, where groups of inputs are paired with groups of outputs, producing a block-diagonal controller structure. For square systems, steady-state tools such as the relative gain array (*RGA*) (Bristol, 1966) and the more versatile block relative gain (*BRG*) (Manousiouthakis et al., 1986) aid in the selection of a control structure, while dynamic measures such as the dynamic block relative gain (*DBRG*) (Arkun, 1987) and the relative sensitivities (Arkun, 1988) contribute to the controller design process.

Processes having an unequal number of inputs and outputs occur frequently in industry, but for control purposes they are often treated as square systems. That is, the system is first "squared" by adding or deleting the appropriate number of inputs or outputs from the system matrix. Consequently, literature dealing with control of a nonsquare system is sparse; indeed, nothing presently exists that addresses the problem of synthesis of nonsquare decentralized control structures. Thus, control of an "unsquared" nonsquare system is often accomplished with a centralized controller.

A nonsquare system may either have more inputs than outputs or more outputs than inputs, with a degenerate case being the square system. Treiber and Hoffman (1986) discuss control of a vacuum distillation unit with five inputs and four outputs, where the system has been squared with a steady-state precompensator. They claim superior performance by the full nonsquare system over that possible when only a square subset of the original system is employed for control. After comparing centralized nonsquare and square control structures in a reactor application study, Morari et al. (1985) concluded that for their system the nonsquare structures are less sensitive to modeling error because they have smaller condition numbers. Nonsquare systems with more outputs than inputs are usually not as desirable because all outputs cannot be controlled exactly at set point as a result of the system being overdetermined. Treiber (1984) considers a distillation column with three inputs that are to keep four outputs at set point in the least-squares sense. A system with more outputs than inputs may also occur if some of the actuators of a square system begin to operate constantly saturated. Grosdidier et al. (1988) show that such cases occur often in practice and the outputs may need to be controlled in the least-squares sense.

Thus nonsquare systems exist in industry and have desirable properties that justify controlling them in their original nonsquare form rather than squaring them. Furthermore, decen-

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tralized control is as advantageous for nonsquare systems as for square systems. Because the analysis tools mentioned above were derived for square systems, they are not immediately useful for nonsquare systems. For these measures to be employed, such systems must first be squared either up or down. As mentioned above, this is usually achieved by either adding or deleting the requisite number of inputs or outputs to obtain a square system matrix. None of these alternatives is desirable. Adding unnecessary outputs to be measured can be costly, while deleting inputs leaves fewer variables to be automatically manipulated in achieving the desired control. Similarly, reducing the number of measured outputs decreases the amount of feedback information available to the system, and arbitrarily adding new manipulated inputs can incur unnecessary cost. Hence, if superior performance can be achieved by implementing decentralized control on the original nonsquare system, this is usually preferable to squaring the system. The present lack of nonsquare analysis tools prevents comparison of the predicted interactions in the nonsquare and squared systems. Therefore, interaction measures directly applicable to nonsquare systems would aid in both the analysis of and the subsequent design of controllers for these systems. This work presents such measures.

Nonsquare Block Relative Gain

The development of a nonsquare block relative gain measure generalizes previous theory to allow analysis of systems with an unequal number of inputs and outputs. Consequently, both the square *BRG* and *RGA* may be considered subcases of the measure to be derived. In the past these square measures could only be used to analyze a nonsquare system after the system had been artificially squared. The nonsquare block relative gain, however, will allow one to work directly with the original nonsquare system matrix. One should note that the name *nonsquare* block relative gain refers to the dimensions of the system to which the measure is applied, rather than to the dimensions of the gain matrix itself. The derivation that follows will show that the nonsquare block relative gain matrix will always be square, whether the system is square or nonsquare. Before we proceed with the derivation of the nonsquare block relative gain we will introduce the essential mathematical background.

Mathematical preliminaries

The following two theorems set forth the properties of the pseudoinverse that will be employed subsequently.

Theorem 1. (Wiberg, 1971) Let $A \in C^{m\times n}$, $b \in C^m$, $x \in C^n$. Consider the equation $Ax = b$. Define $x_0 = A^\dagger b$. Then $\|Ax - b\|_2 \geq \|Ax_0 - b\|_2$ and for those $x \neq x_0$ such that $\|Ax - b\|_2 = \|Ax_0 - b\|_2$, then $\|x\|_2 > \|x_0\|_2$.

That is, if no solution to the equation $Ax = b$ exists, $A^\dagger b$ gives the closest possible, or least-squares, solution. If the solution to the equation $Ax = b$ is not unique, then $A^\dagger b$ is the solution with minimum norm.

Theorem 2. (Graybill, 1969) Let $A \in C^{m\times n}$.
(*i*) If $A \in C_m^{m\times n}$, then $A^\dagger = A^H(AA^H)^{-1}$ and $AA^\dagger = I$.
(*ii*) If $A \in C_n^{m\times n}$, then $A^\dagger = (A^HA)^{-1}A^H$ and $A^\dagger A = I$.

Therefore this theorem establishes simple expressions for the pseudoinverse of a matrix with full rank, where $C_r^{p\times q}$ denotes a complex $p \times q$ matrix of rank r and the superscript H indicates the Hermitian transpose of a matrix. (Note therefore that $A \in C_m^{m\times n}$ will imply that the system has more inputs than outputs, while $A \in C_n^{m\times n}$ will imply that there are more outputs than inputs.)

Definition of a nonsquare block relative gain

Consider the open-loop transfer function matrix of a nonsquare plant, $G(s) \in C^{m\times n}$ (the Laplace variable s will subsequently be dropped) with the following 2×2 block partitioning:

$$G = \begin{matrix} \updownarrow m_1 \\ \updownarrow m_2 \end{matrix} \overset{\quad\leftarrow n_1 \rightarrow \quad \leftarrow n_2 \rightarrow}{\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}} \tag{1}$$

with

$$\begin{matrix} \updownarrow m_1 \\ \updownarrow m_2 \end{matrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = G \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix} \tag{2}$$

The partitioning of G will always be restricted as follows: If G has more outputs than inputs, then its diagonal blocks must either be square or have more outputs than inputs. Conversely, if G has more inputs than outputs, its diagonal blocks must have the number of inputs greater than or equal to the number of outputs. Furthermore, G and its diagonal blocks are assumed to be of full rank such that theorem 2 applies to each block.

In the original formulation of a block relative gain measure, Manousiouthakis et al. (1986) showed that both a BRG_l and a BRG_r could be derived for a square system, based on the order of matrix multiplication. However, only the BRG_l related to the performance of the system. Furthermore, it was the inverse of the left block relative gain, BRG_l^{-1}, that appeared in the performance equations. The development that follows will show that these conditions also hold in the case of a nonsquare system. Thus the BRG_l^{-1} is the critical measure to be employed in evaluating block decentralized structures. Note that the BRG_l^{-1} is itself a block relative gain measure—the inverse merely denotes that it is the inverse of the block relative gain as *originally defined*. Indeed, it would now seem that a better definition for the block relative gain (and the scalar relative gain) would be the *inverse* of the original definition. As will become evident later in the development, such a definition is not only preferable but is necessary for the nonsquare case. Thus the reader should note that the block relative gain discussed in this paper will be denoted BRG_l^{-1} and will represent the inverse of the traditionally defined block relative gain.

The definition of the block relative gain for square systems, expressed now as the inverse of the traditional definition, is therefore:

$$BRG_{l_{11}}^{-1} = \left(\left[\frac{\partial y_1}{\partial u_1} \bigg|_{u_2=0} \right] \cdot \left[\frac{\partial y_1}{\partial u_1} \bigg|_{y_2=0} \right]^{-1} \right)^{-1} = \left[\frac{\partial y_1}{\partial u_1} \bigg|_{y_2=0} \right] \cdot \left[\frac{\partial y_1}{\partial u_1} \bigg|_{u_2=0} \right]^{-1} \tag{3}$$

That is, the block relative gain serves as a ratio of the open-loop gain of the first subsystem when the second subsystem is kept under "perfect" control and the open-loop gain of the first subsystem when the second subsystem is not under control. This definition obviously cannot be applied to a nonsquare system because of the presence of the inverse term. A more general definition capable of handling systems with more inputs than outputs, those with more outputs than inputs, and square systems is required. Such a definition is obtained when the pseudoinverse is utilized. Therefore the following definition will be employed in subsequent references to the block relative gain:

$$BRG_{l_{11}}^{-1} = \left[\frac{\partial y_1}{\partial u_1}\bigg|_{y_2=0}\right] \cdot \left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_2=0}\right]^{\dagger} \tag{4}$$

Note that this expression reduces to Eq. 3 when the system is square.

Expressions of the nonsquare BRG^{-1}

The above definition is applied to the system equations:

$$y_1 = G_{11}u_1 + G_{12}u_2 \tag{5}$$

$$y_2 = G_{21}u_1 + G_{22}u_2 \tag{6}$$

to obtain the two open-loop gains:

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_2=0}\right] = G_{11} \tag{7}$$

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{y_2=0}\right] = G_{11} - G_{12}[G_{22}]^{\dagger}G_{21} \tag{8}$$

Note that for the case of more outputs than inputs, perfect control in the second subsystem will be an imposed condition, producing an equation for which only a least-squares solution may be possible. That solution leads to the above expression for the open-loop gain when perfect control is imposed.

Therefore, for a system with *more inputs than outputs* ($G_{11} \in C_{m_1}^{m_1 \times n_1}$), the resulting block relative gain is:

$$BRG_{l_{11}}^{-1} = I_{m_1} - G_{12}[G_{22}]^{\dagger}G_{21}[G_{11}]^{\dagger} \tag{9}$$

and for a system with *more outputs than inputs* ($G_{11} \in C_{n_1}^{m_1 \times n_1}$), the proper expression is:

$$BRG_{l_{11}}^{-1} = G_{11}[G_{11}]^{\dagger} - G_{12}[G_{22}]^{\dagger}G_{21}[G_{11}]^{\dagger} \tag{10}$$

One should observe that the $BRG_{l_{11}}^{-1}$ for a system with more outputs than inputs will be singular. Thus its inverse, the traditionally defined left block relative gain, will not exist for this case. Nevertheless the BRG_l^{-1} will exist and will be shown to relate to system performance through the dynamic block relative gain. It should be noted that the BRG^{-1} for a nonsquare G_{11} will be *square:* $BRG_{l_{11}}^{-1} \in C^{m_1 \times m_1}$. Furthermore, when $G_{11} = G$ (i.e., no decentralization), the block relative gain will be the identity matrix for a system with more inputs than outputs. A system with more outputs than inputs and $G_{11} = G$ will have a block relative gain of $GG^{\dagger}$, its least-squares approximation to the identity matrix. Note also that Eqs. 9 and 10 reduce to the square BRG^{-1} expression for the case of $G \in C^{n \times n}$.

When working with a square system matrix, further partitioning of the system into more than 2 × 2 blocks does not alter the form of the BRG^{-1} expressions for the individual blocks. However, for the nonsquare case, BRG^{-1} expressions different from Eqs. 9 and 10 are required for systems partitioned into more than 2 × 2 blocks. To obtain the proper expression, first consider the open-loop transfer function matrix of a nonsquare plant, $G \in C^{m \times n}$ with the following $k \times k$ block partitioning:

$$G = \begin{array}{c} \updownarrow m_1 \\ \updownarrow m_2 \\ \vdots \\ \updownarrow m_k \end{array} \overset{\quad \leftarrow n_1 \rightarrow \quad \leftarrow n_2 \rightarrow \quad \cdots \quad \leftarrow n_k \rightarrow}{\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1k} \\ G_{21} & G_{22} & \cdots & G_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k1} & G_{k2} & \cdots & G_{kk} \end{bmatrix}} \triangleq \begin{array}{c} \updownarrow m_1 \\ \updownarrow m - m_1 \end{array} \overset{\leftarrow n_1 \rightarrow \quad \leftarrow n - n_1 \rightarrow}{\begin{bmatrix} G_{11} & G_{cr_{11}} \\ G_{cd_{11}} & G_{c_{11}} \end{bmatrix}} \tag{11}$$

The block model of this plant is then:

$$G_b = \begin{bmatrix} G_{11} & & & \\ & G_{22} & & \\ & & \ddots & \\ & & & G_{kk} \end{bmatrix} \triangleq \begin{bmatrix} G_{11} & \\ & G_{b_{11}} \end{bmatrix} \tag{12}$$

The system structure is therefore described by the G_{11} block and its complement, consisting of the $k - 1$ remaining blocks.

The proper definition of the left block relative gain for a $k \times k$ system parallels the 2 × 2 definition:

$$BRG_{l_{11}}^{-1} = \left[\frac{\partial y_1}{\partial u_1}\bigg|_{y_i=0, i\neq 1}\right] \cdot \left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_i=0, i\neq 1}\right]^{\dagger} \tag{13}$$

As in the 2 × 2 case, the derivation of the proper BRG^{-1} expressions begins with evaluation of the open-loop gains:

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_i=0, i\neq 1}\right] = G_{11} \tag{14}$$

Two different cases must be considered when evaluating the open-loop gain with all other subsystems under perfect control. (Refer to the supplementary material for the derivations of Eqs. 15 and 16). Consider first the system with $G_{11} \in C_{m_1}^{m_1 \times n_1}$:

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{y_i=0, i\neq 1}\right] = G_{11} - G_{cr_{11}}[G_{b_{11}}]^{\dagger}(G_{c_{11}}[G_{b_{11}}]^{\dagger})^{-1}G_{cd_{11}} \tag{15}$$

For the case where $G_{11} \in C_{n_1}^{m_1 \times n_1}$, perfect control is again an imposed condition that now produces a system of equations for which only least-squares solutions are possible. Those solutions lead to the following expression:

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{y_i=0, i\neq 1}\right] = G_{11} - G_{cr_{11}}([G_{b_{11}}]^{\dagger}G_{c_{11}})^{-1}[G_{b_{11}}]^{\dagger}G_{cd_{11}} \tag{16}$$

These derivations are valid for any block G_{ii} after the blocks within G_b are rearranged to obtain the appropriate $G_{c_{ii}}$, $G_{cd_{ii}}$, etc. Thus the proper block relative gain expression for $G_{ii} \in C_{m_i}^{m_i \times n_i}$

is:

$$BRG_{I_{ii}}^{-1} = I_{m_i} - G_{cr_{ii}}[G_{b_{ii}}]^{\dagger}(G_{c_{ii}}[G_{b_{ii}}]^{\dagger})^{-1}G_{cd_{ii}}[G_{ii}]^{\dagger} \quad (17)$$

Similarly, the proper expression for $G_{ii} \in C_{n_i}^{m_i \times n_i}$ is:

$$BRG_{I_{ii}}^{-1} = G_{ii}[G_{ii}]^{\dagger} - G_{cr_{ii}}([G_{b_{ii}}]^{\dagger}G_{c_{ii}})^{-1}[G_{b_{ii}}]^{\dagger}G_{cd_{ii}}[G_{ii}]^{\dagger} \quad (18)$$

The above equations also apply to the case of a square system matrix. For a square system, however, exact solutions rather than least-squares or minimum-norm solutions are obtained for each equation in the derivation. This is because the pseudoinverses reduce to inverses. Consequently, for a square system, the final BRG^{-1} expressions 17 and 18 reduce to the simpler 2×2 forms of Eqs. 9 and 10 with pseudoinverses replaced by inverses without loss of accuracy. That is, the block structure of the complement block $G_{c_{ii}}$ does not affect the BRG^{-1} expressions when the system matrix is square. This is not the case for a nonsquare system, making Eqs. 17 and 18 the only BRG^{-1} expressions that may be used for a system larger than 2×2 blocks.

Note that BRG^{-1} depends on the Laplace variable s (or the frequency ω), which has been omitted throughout the above development. Therefore, if perfect control were to hold at all frequencies, $BRG^{-1}(\omega)$ would constitute a dynamic interaction measure. However, since for $G \in C_m^{m \times n}$ perfect control can only be achieved by integral action at steady state, and for $G \in C_n^{m \times n}$ perfect control is best approximated by integral action at steady state, BRG^{-1} will be evaluated at $s = 0$ and used as a steady-state interaction measure. Because the block relative gain was defined as a ratio of open-loop gains, the ideal value of the BRG^{-1} will be the identity matrix. Note that this value is not achievable for $G \in C_n^{m \times n}$. Instead, the best attainable, or ideal, BRG^{-1} will be $G_{ii}[G_{ii}]^{\dagger}$, which is the least-squares approximation to the identity matrix for a given subsystem.

Properties of the nonsquare BRG $^{-1}$

The block relative gain is defined relative to the first m_1 outputs and first n_1 inputs of the plant matrix G. Let G be partitioned into k blocks as in Eq. 11.

Consider a permutation $G' = P_1 G P_2$ where P_1 and P_2 are permutation matrices with the following special structure:

$$G' = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{12} \end{bmatrix} G \begin{bmatrix} P_{21} & 0 \\ 0 & P_{22} \end{bmatrix} \quad (19)$$

Then the following results hold (proofs are given in the supplementary material.):

Theorem 3. For a chosen G_b consisting of blocks having fixed sets of inputs and outputs, the sets of diagonal elements of $(BRG_{I_{11}}^{-1})'$ remain the same for all possible rearrangements of inputs and outputs.

Theorem 4. For a chosen block model G_b with blocks having fixed sets of inputs and outputs, the eigenvalues $\lambda_i[(BRG_{I_{11}}^{-1})']$ remain the same for all possible rearrangements of inputs and outputs.

Theorem 5. The block relative gain of G'_{11} is as follows:

$$(BRG_{I_{11}}^{-1})' = P_{11} \cdot BRG_{I_{11}}^{-1} \cdot P_{11}^T \quad (20)$$

Theorems 3 and 4 reveal the utility of both the diagonal elements and the eigenvalues of the BRG^{-1} in screening possible control structures. The desired value for the diagonal elements and eigenvalues of the $BRG_{I_{11}}^{-1}$ for $G_{11} \in C_{m_1}^{m_1 \times n_1}$ will be 1.0 because the ideal $BRG_{I_{11}}^{-1}$ is the identity matrix. These theorems show that for a chosen $G_{11} \in C_{m_1}^{m_1 \times n_1}$ in a specified G_b, if these values are not sufficiently close to 1.0, as desired for a noninteractive system, further rearrangement within blocks will not alter the set of values that can be achieved. For $G_{11} \in C_{n_1}^{m_1 \times n_1}$ the ideal $BRG_{I_{11}}^{-1}$ will be the matrix $G_{11}[G_{11}]^{\dagger}$. We know from algebra that for nonsquare matrices A and B, the matrices AB and BA will share the same eigenvalues, but the larger matrix will have additional eigenvalues which are equal to zero. Since $[G_{11}]^{\dagger}G_{11} = I$ for $G_{11} \in C_{n_1}^{m_1 \times n_1}$ this means that $G_{11}[G_{11}]^{\dagger}$ will only have eigenvalues of 0 and 1.0. Note that because the $BRG_{I_{11}}^{-1}$ for a $G_{11} \in C_{n_1}^{m_1 \times n_1}$ will always be singular, it will have at least one zero eigenvalue. Thus, for this case, if the *nonzero* eigenvalues of $BRG_{I_{11}}^{-1}$ are not close enough to 1.0 or the diagonal elements of $BRG_{I_{11}}^{-1}$ do not approach those of the matrix $G_{11}[G_{11}]^{\dagger}$, further rearrangement within blocks will not alter the achievable set of values. Indeed, theorem 5 reveals that the $(BRG^{-1})'$ of the permuted G'_{11} is simply a rearrangement of the original BRG^{-1}. These results significantly affect the dimensionality of the structure-selection process. That is, for any G_b consisting of blocks having fixed sets of inputs and outputs, only one BRG^{-1} need be calculated for the designated G_{11}. Any other arrangement of the inputs and outputs *within* blocks would produce a $(BRG^{-1})'$ conveying no new information.

Several properties of the square BRG^{-1} do not carry over to the nonsquare measure. A relationship between the diagonal elements of a BRG^{-1} and its component one-dimensional BRG^{-1}'s similar to that presented by Manousiouthakis et al. (1986) for square systems has not been found to exist for the nonsquare case. Furthermore, the nonsquare BRG^{-1} can only be considered to be scaling invariant under the trivial case of scaling all inputs and/or outputs by the same scalar value.

Nonsquare Dynamic Block Relative Gain

While the nonsquare block relative gain was derived by imposing a perfect control constraint, the dynamic block relative gain ($DBRG^{-1}$) relaxes this assumption. Consequently, these new expressions will depend on the controller chosen and may therefore be employed as a tool in the process of controller design.

Derivation of the nonsquare DBRG $^{-1}$

As was done for the BRG^{-1}, the nonsquare $DBRG^{-1}$ will first be derived for the case of 2×2 block partitioning so that the principles involved may be more easily observed.

The internal model control (IMC) system depicted in Figure 1 will be employed for the derivation of the dynamic measure. Because perfect control will not be assumed, the BRG^{-1} definition must be altered to be the following:

$$DBRG_{I_{11}}^{-1} = \left[\frac{\partial y_1}{\partial u_1}\bigg|_{r_2=0}\right] \cdot \left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_2=0}\right]^{\dagger} \quad (21)$$

Thus, control is described by $r_2 = 0$ rather than the perfect control condition, $y_2 = 0$.

As with the BRG^{-1}, the open-loop gains serve as the starting

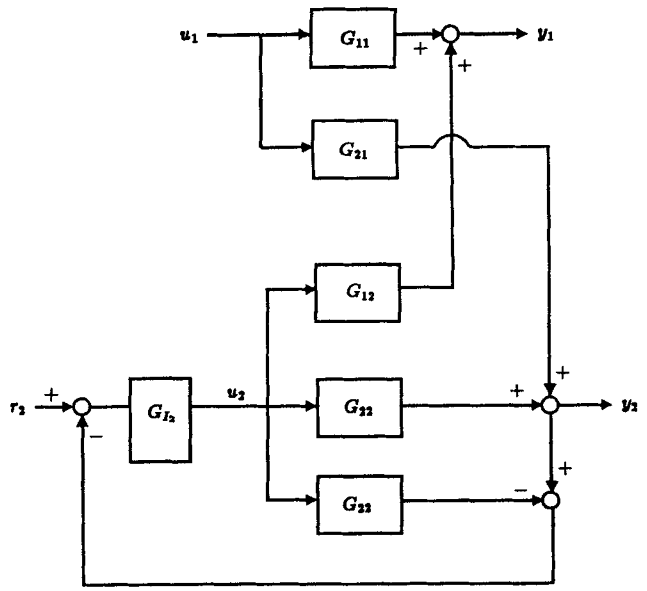

Figure 1. IMC for second subsystem of a 2 × 2 block system.

point of the $DBRG^{-1}$ derivation:

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_2=0}\right] = G_{11} \tag{22}$$

$$\left[\frac{\partial y_1}{\partial u_1}\bigg|_{r_2=0}\right] = G_{11} - G_{12}[G_{22}]^{\dagger}H_2G_{21} \tag{23}$$

where $H_2 = G_{22}G_{I_2}$ is defined to be the desired closed-loop performance of the second subsystem if it were isolated from the first. That is, H_2 is chosen by the designer, and the corresponding controller is obtained by solving for $G_{I_2} = [G_{22}]^{\dagger}H_2$. Note that for a subsystem with more inputs that outputs, this solution means that the desired performance will be achievable: $G_{22}G_{I_2} = G_{22}[G_{22}]^{\dagger}H_2 = H_2$. However, for a subsystem with more outputs than inputs, the controller is obtained by a least-squares solution, and the desired performance is therefore *not* achievable: $G_{22}G_{I_2} = G_{22}[G_{22}]^{\dagger}H_2 \neq H_2$:

The resulting dynamic block relative gain expression for $G_{11} \in C_{m_1}^{m_1 \times n_1}$ is:

$$DBRG_{I_{11}}^{-1} = I_{m_1} - G_{12}[G_{22}]^{\dagger}H_2G_{21}[G_{11}]^{\dagger} \tag{24}$$

and for $G_{11} \in C_{n_1}^{m_1 \times n_1}$ is:

$$DBRG_{I_{11}}^{-1} = G_{11}[G_{11}]^{\dagger} - G_{12}[G_{22}]^{\dagger}H_2G_{21}[G_{11}]^{\dagger} \tag{25}$$

It is important to note that for a square system matrix the $DBRG_{I_{11}}^{-1}$ equation reduces to the inverse of the $DBRG_{11}$ equation as presented by Arkun (1987). Furthermore, the above expressions reduce to the BRG^{-1} expressions 9 and 10 when perfect control is imposed ($H_2 = I$). As with the BRG^{-1}, although the system matrix is nonsquare, a $DBRG_{I_{11}}^{-1} \in C^{m_1 \times m_1}$ will be produced.

Further partitioning of the system G into more than 2 × 2 blocks necessitates a more general $DBRG^{-1}$ expression. Consider the system G and its block model G_b of Eqs. 11 and 12. The controller structure for the subsystems in the complement of G_{11} being controlled by IMC will be:

$$G_{I_{c11}} = \begin{bmatrix} G_{I_2} & & \\ & \ddots & \\ & & G_{I_k} \end{bmatrix} = [G_{b_{11}}]^{\dagger} \cdot bd(H_i)_{i \neq 1} \tag{26}$$

where $G_{I_{c11}} \in C^{(n-n_1)\times(m-m_1)}$, $G_{I_i} \in C^{n_i \times m_i}$, and $bd(H_i)_{i\neq 1}$ is the block-diagonal matrix of H_i, $i \neq 1$.

The $DBRG_{I_{11}}^{-1}$ for $k \times k$ block partitioning is defined to be:

$$DBRG_{I_{11}}^{-1} = \left[\frac{\partial y_1}{\partial u_1}\bigg|_{r_i=0, i\neq 1}\right] \cdot \left[\frac{\partial y_1}{\partial u_1}\bigg|_{u_i=0, i\neq 1}\right]^{\dagger} \tag{27}$$

This definition holds for any subsystem i if the system matrix G is appropriately rearranged. Therefore the dynamic block relative gain expression for $G_{ii} \in C_{m_i}^{m_i \times n_i}$ will be (see the supplementary material for details):

$$DBRG_{I_{ii}}^{-1} = I_{m_i} - G_{cr_{ii}}[G_{b_{ii}}]^{\dagger}bd(H_j)_{j\neq i} \cdot (I_{m-m_i} + (G_{c_{ii}} - G_{b_{ii}})[G_{b_{ii}}]^{\dagger}bd(H_j)_{j\neq i})^{-1}G_{cd_{ii}}[G_{ii}]^{\dagger} \tag{28}$$

Similarly, the proper expression for $G_{ii} \in C_{n_i}^{m_i \times n_i}$ is:

$$DBRG_{I_{ii}}^{-1} = G_{ii}[G_{ii}]^{\dagger} - G_{cr_{ii}}[G_{b_{ii}}]^{\dagger}bd(H_j)_{j\neq i} \cdot (I_{m-m_i} + (G_{c_{ii}} - G_{b_{ii}})[G_{b_{ii}}]^{\dagger}bd(H_j)_{j\neq i})^{-1}G_{cd_{ii}}[G_{ii}]^{\dagger} \tag{29}$$

The $DBRG_{I_{ii}}^{-1}$ equations, when applied to a square system matrix, reduce to that derived by Arkun (1987). Furthermore, when perfect control is imposed on the $k - 1$ subsystems, $bd(H_j)_{j\neq i} = I_{m-m_i}$ and the $DBRG^{-1}$ Eqs. 28 and 29 become the BRG^{-1} Eqs. 17 and 18 presented earlier.

Connections to closed-loop performance and stability

Consider the closed-loop IMC system depicted in Figure 2. The performance equation for the IMC system, whether G is square or nonsquare, is as follows:

$$y = GG_I[I_m + (G - G_b)G_I]^{-1}r = Yr \tag{30}$$

The expressions for G and G_b given by Eqs. 11 and 12, respectively, will apply. The controller matrix will be defined as:

$$G_I = \begin{bmatrix} G_{I_1} & \\ & G_{I_{c11}} \end{bmatrix} \tag{31}$$

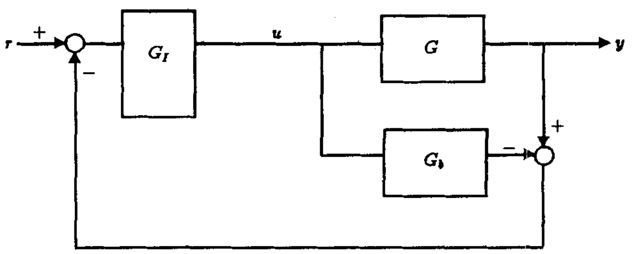

Figure 2. Decentralized IMC for a nonsquare system.

where $G_{I_{c_{ii}}}$ is defined by Eq. 26. By substituting for these expressions in Eq. 30 and employing the definition of the inverse of a block matrix given in Kailath (1980), the resulting local performance equation for y_i (i.e., the response of y_i to a change in its own set point, r_i) is produced. (For details, see the supplementary material.) For $G_{ii} \in C_{m_i}^{m_i \times n_i}$ (a system with more manipulated inputs than outputs) the response may be expressed in terms of the $DBRG_{I_{ii}}^{-1}$ as follows:

$$Y_{ii} = DBRG_{I_{ii}}^{-1} H_i[I_{m_i} + (DBRG_{I_{ii}}^{-1} - I_{m_i})H_i]^{-1} \quad (32)$$

while for $G_{ii} \in C_{n_i}^{m_i \times n_i}$ (a system with fewer manipulated inputs than outputs) the response is related to the $DBRG_{I_{ii}}^{-1}$ in the following manner:

$$Y_{ii} = DBRG_{I_{ii}}^{-1} H_i[I_{m_i} + (DBRG_{I_{ii}}^{-1} - G_{ii}[G_{ii}]^{\dagger})H_i]^{-1} \quad (33)$$

Note that in the latter case all of the outputs cannot be kept at their set points since there will be fewer manipulated inputs than controlled outputs. Instead, because the controllers are obtained using least-squares solutions as described previously, the outputs will be controlled in the least-squares sense.

When $DBRG_{I_{ii}}^{-1}(s) = I_{m_i}$ for $G_{ii} \in C_{m_i}^{m_i \times n_i}$ or $DBRG_{I_{ii}}^{-1}(s) = G_{ii}[G_{ii}]^{\dagger}$ for $G_{ii} \in C_{n_i}^{m_i \times n_i}$, the corresponding local closed-loop performance is not affected by dynamic interactions. That is, for $G_{ii} \in C_{m_i}^{m_i \times n_i}$, $Y_{ii} = H_i = G_{ii}G_{I_i}$, while for $G_{ii} \in C_{n_i}^{m_i \times n_i}$, $Y_{ii} = G_{ii}[G_{ii}]^{\dagger}H_i = G_{ii}G_{I_i}$ will be the physical response. This establishes the relationship between $DBRG^{-1}$ and the closed-loop performance. Similarly to the square case (Arkun, 1987), $\sigma^*(DBRG_{I_{ii}}^{-1}(j\omega) - I)$ for $G_{ii} \in C_{m_i}^{m_i \times n_i}$ and $\sigma^*(DBRG_{I_{ii}}^{-1}(j\omega) - G_{ii}[G_{ii}]^{\dagger})$ for $G_{ii} \in C_{n_i}^{m_i \times n_i}$ represent the magnitudes of closed-loop dynamic interactions, where σ^* denotes the maximum singular value. If this quantity is small over a large frequency bandwidth, better performance with a higher bandwidth for Y_{ii} can be achieved by keeping $H_i(\omega)$ closer to I.

At steady state, if H_i is chosen to provide integral action in the controller, H_i will be I. Thus Eq. 32 confirms that, for a square system or one with more inputs than outputs, Y_{ii} will be I at steady state. Note, however, that such a result is not obtained from Eq. 33. Because there are fewer manipulated variables than controlled variables, even integral action ($H_i = I$) will not bring all of the outputs to set point at steady state. Instead the following expression for the steady-state performance is produced:

$$Y_{ii}(0) = BRG_{I_{ii}}^{-1}(0)\{I_{m_i} + [BRG_{I_{ii}}^{-1}(0) - G_{ii}[G_{ii}]^{\dagger}]\}^{-1} \quad (34)$$

This equation reveals that for a decentralized system with fewer inputs than outputs, the steady-state performance of each subsystem depends on the value of its block relative gain. Thus for such a system, the proper selection of a decentralized structure using the BRG^{-1} is critical not only to the dynamic performance of the system, but also to its ultimate steady-state performance.

The following theorem relates the $DBRG^{-1}$ to closed-loop stability for nonsquare systems.

Theorem 6. Assume that $G_{ij}(s)$'s and G_{I_i}'s are stable. Then if $[I_{m-m_i} + (G_{c_{ii}} - G_{b_{ii}})[G_{b_{ii}}]^{\dagger}bd(H_j)_{j\neq i}]^{-1}$ is stable for $i = 1, 2, \ldots, k$, the closed-loop system is stable if and only if the net sum of encirclements of $(-1, 0)$ by the characteristic loci

(i) $\lambda_i[(DBRG_{I_{ii}}^{-1} - I_{m_i})H_i]$ for $G \in C_m^{m\times n}$, or

(ii) $\lambda_i[(DBRG_{I_{ii}}^{-1} - G_{ii}[G_{ii}]^{\dagger})H_i]$ for $G \in C_n^{m\times n}$

is zero for $i = 1, 2, \ldots, k$

Proof. See supplementary material.

Therefore the $DBRG^{-1}$ allows the designer to draw conclusions about the stability of the overall nonsquare system from the stability results of the individual subsystems. One should note that requiring $[I_{m-m_i} + (G_{c_{ii}} - G_{b_{ii}})[G_{b_{ii}}]^{\dagger}bd(H_j)_{j\neq i}]^{-1}$ to be stable for $i = 1, 2, \ldots, k$ is equivalent to requiring closed-loop stability from the complementary subsystem $G_{c_{ii}}$. That is, the decentralized system must be fault tolerant with respect to failure of any subsystem i. Thus, while this condition introduces conservatism into the stability theorem, it represents a very desirable physical property for the decentralized system to possess. In other words, while there may exist controllers that do not meet the above condition but still stabilize the closed-loop system, those controllers will not produce a closed-loop system that is fault tolerant in the manner described above.

Relative Sensitivity for Nonsquare Systems

The block relative gain and dynamic block relative gain have been shown to relate to the local closed-loop performances of a system. That is, BRG^{-1} and $DBRG^{-1}$ carry information about how a subsystem will respond to changes in its own set point. But a subsystem that does not see interactions in its local performance may see significant interactions from a set-point change in another subsystem. For example, a *triangular* square system will automatically produce ideal-valued BRG^{-1}'s and $DBRG^{-1}$'s, but one-way interactions will still exist. Even more interesting is the fact that some *full* nonsquare systems will produce ideal-valued BRG^{-1}'s and $DBRG^{-1}$'s, a desirable result not possible with square systems. However, such systems will still exhibit one-way interactions that need to be quantified.

Arkun (1988) has introduced the relative sensitivity measure to describe the direction and magnitude of such cross interactions for square systems. The measure conveys both controller-independent information and controller-dependent information. Hence, application of the relative sensitivity as an interaction measure to nonsquare systems should be an integral step in both the structure selection and controller design processes.

Derivation of a nonsquare relative sensitivity

Consider the $k \times k$ IMC system depicted in Figure 2. The relative sensitivity between any two subsystems is defined by:

$$S_{ji} = \left[\frac{\partial y_j}{\partial r_i}\right]\left[\frac{\partial y_i}{\partial r_i}\right]^{-1} = Y_{ji}Y_{ii}^{-1} \quad i, j = 1, 2, \ldots, k \quad (35)$$

At this point it must be noted that this definition restricts the application of the relative sensitivity measure to systems such that $G \in C_m^{m\times n}$. This is because the inverse of the local performance Y_{ii} does not exist for systems with $G \in C_n^{m\times n}$. (Such systems will have $Y_{ii} \in C_{n_i}^{m_i \times m_i}$ where $n_i < m_i$. Therefore Y_{ii} will be singular.) Hence the subsequent discussion will only relate to systems with more inputs than outputs.

This definition of the relative sensitivity leads to a relative sensitivity matrix, S, consisting of the elements S_{ji}:

$$S = \begin{bmatrix} I & S_{12} & \cdots & S_{1k} \\ S_{21} & I & \cdots & S_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ S_{k1} & S_{k2} & \cdots & I \end{bmatrix} \quad (36)$$

Note that each column of the S matrix is formed by the relative sensitivity between each subsystem G_{ii} and its complement $G_{c_{ii}}$:

$$S_{c_{ii}i} \triangleq Y_{cd_{ii}} Y_{ii}^{-1} = \begin{bmatrix} Y_{1i}Y_{ii}^{-1} \\ \vdots \\ Y_{ji}Y_{ii}^{-1} \\ \vdots \\ Y_{ki}Y_{ii}^{-1} \end{bmatrix}_{j \neq i} \triangleq \begin{bmatrix} S_{1i} \\ \vdots \\ S_{ji} \\ \vdots \\ S_{ki} \end{bmatrix}_{j \neq i} \quad (37)$$

The relative sensitivity between G_{ii} and $G_{c_{ii}}$ may be expressed as (see the supplementary material for details):

$$S_{c_{ii}i} = -bd(H_j - I_{m_j})_{j \neq i} \cdot \{I_{m-m_i} + [(G_{c_{ii}} - G_{b_{ii}}) - G_{cd_{ii}}[G_{ii}]^{\dagger} G_{cr_{ii}}][G_{b_{ii}}]^{\dagger} bd(H_j)_{j \neq i}\}^{-1} \cdot G_{cd_{ii}}[G_{ii}]^{\dagger} \quad (38)$$

After observing that $S_{ii} = I$, this value may be inserted into the appropriate position of each column to complete the S matrix. Therefore by applying the $S_{c_{ii}i}$ equation k times, the S matrix is evaluated. Note that for the case of $k = 2$, there will be two relative sensitivities with the following simplified expressions:

$$S_{12} = -(H_1 - I_{m_1})\{I_{m_1} + [BRG_{I_{11}}^{-1}(j\omega) - I_{m_1}]H_1\}^{-1} G_{12}[G_{22}]^{\dagger} \quad (39)$$

$$S_{21} = -(H_2 - I_{m_2})\{I_{m_2} + [BRG_{I_{22}}^{-1}(j\omega) - I_{m_2}]H_2\}^{-1} G_{21}[G_{11}]^{\dagger} \quad (40)$$

Connections with closed-loop performance

By rearranging Eq. 37, the closed-loop interactions may be expressed as a function of the relative sensitivity: $Y_{cd_{ii}} = S_{c_{ii}i} \cdot Y_{ii}$ for $i = 1, 2, \ldots, k$. Therefore, to minimize the magnitude of the interactions, it will be desirable to minimize the magnitude of $S_{c_{ii}i}$, particularly for the bandwidth that the magnitude of Y_{ii} will be close to one. Because the relative sensitivities depend on controller tuning, this provides another guideline for controller design, which may be employed in conjunction with those obtained from $DBRG^{-1}$.

At steady state, $H_i = I$ and all S_{ji} will be zero. For high frequencies $H_i = 0$ and the relative sensitivity will approach an asymptote:

$$S_{c_{ii}i}(\omega \rightarrow \infty) \triangleq \bar{S}_{c_{ii}i}(\omega) = G_{cd_{ii}}[G_{ii}]^{\dagger} = \begin{bmatrix} G_{1i}[G_{ii}]^{\dagger} \\ \vdots \\ G_{ji}[G_{ii}]^{\dagger} \\ \vdots \\ G_{ki}[G_{ii}]^{\dagger} \end{bmatrix}_{j \neq i} \triangleq \begin{bmatrix} \bar{S}_{1i} \\ \vdots \\ \bar{S}_{ji} \\ \vdots \\ \bar{S}_{ki} \end{bmatrix}_{j \neq i} \quad (41)$$

Therefore a relative sensitivity asymptote matrix may be constructed:

$$\bar{S} \triangleq \begin{bmatrix} I & \bar{S}_{12} & \cdots & \bar{S}_{1k} \\ \bar{S}_{21} & I & \cdots & \bar{S}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{S}_{k1} & \bar{S}_{k2} & \cdots & I \end{bmatrix} = \begin{bmatrix} I & G_{12}[G_{22}]^{\dagger} & \cdots & G_{1k}[G_{kk}]^{\dagger} \\ G_{21}[G_{11}]^{\dagger} & I & \cdots & G_{2k}[G_{kk}]^{\dagger} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k1}[G_{11}]^{\dagger} & G_{k2}[G_{22}]^{\dagger} & \cdots & I \end{bmatrix} \quad (42)$$

Because these asymptotes do not depend on the controllers, they are a dynamic tool to be used for structure selection. It will be desirable to have structures that minimize the magnitude of the asymptotes. Specifically, asymptotes whose magnitudes exceed one will not be desirable, for this will lead to large Y_{ji} interaction terms.

Summary of Results

A block relative gain measure for nonsquare systems has been derived. Evaluated at steady state, this measure serves as a performance-related tool for evaluating control structures prior to controller design. The relative sensitivity asymptotes have also been derived as a dynamic tool for screening alternative control structures. Both measures are independent of controller tuning and are related to closed-loop performance. The block relative gain yields information about interactions in the local performance of a subsystem while the relative sensitivity asymptotes provide insight about the direction and magnitude of interactions between subsystems.

Both the $DBRG^{-1}$ and the relative sensitivities have been presented as dynamic interaction measures that depend on the controller tuning and are consequently applicable to controller design. Note that these quantities assess the *actual* interactions of a closed-loop system while the BRG^{-1} and the relative sensitivity asymptotes only accurately measure the interactions present in such limiting cases as steady state and high frequencies.

Example

Consider a mixing tank that has three input streams and one exit stream. The variables to be controlled are the height of liquid in the tank, h, and the exit concentration, c_A. The concentrations of the input streams will be fixed and the exit flow rate will be set by the liquid height. Hence, the variables available for manipulation are the flow rates of the input streams, F_1, F_2, and F_3. Once the mass balances have been linearized around a chosen steady state and cast in deviation variables, a transfer function matrix may be calculated:

$$G(s) = \begin{bmatrix} \frac{4}{20s+1} & \frac{4}{20s+1} & \frac{4}{20s+1} \\ \frac{3}{10s+1} & \frac{-3}{10s+1} & \frac{5}{10s+1} \end{bmatrix} \quad (43)$$

The outputs for this system will be y_1 and y_2, corresponding to h and c_A, respectively. Similarly, the input flow rates F_1, F_2, and F_3 will be represented by the three inputs u_1, u_2, and u_3.

Structure selection using BRG^{-1} *and relative sensitivity asymptotes*

For this system it may be preferable to control the two outputs by only manipulating two of the three inputs available. That is, if satisfactory performance results from utilizing only two manipulated variables, the third will not be employed. Therefore the *RGA*'s of the component squared systems are examined first. Manipulating only u_1 and u_2 to control y_1 and y_2 yields the following relative gain array:

$$RGA_{12} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad (44)$$

This is obviously unsatisfactory for it suggests that neither of the alternatives for pairing the variables is very good. The system that results when only u_1 and u_3 are manipulated to control y_1 and y_2 has the following relative gain array:

$$RGA_{13} = \begin{bmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{bmatrix} \quad (45)$$

This option is also undesirable, for the *RGA* elements indicate significant interactions that should result in poor performance. The last component square system uses u_2 and u_3 for control of y_1 and y_2. This combination produces the following *RGA*:

$$RGA_{23} = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix} \quad (46)$$

The indicated pairing here is (y_1, u_2) and (y_2, u_3). The *RGA* suggests that the interactions with this pairing should be slightly smaller than those which would occur in the first component square system. This last squared system is therefore the best SISO pairing possible with this set of inputs and outputs.

The nonsquare system should now be examined for possible improvement of the performance.

The nonsquare system contains six possible decentralized control structures. Examination of the BRG^{-1}'s for each structure reveals that two of them produce identity BRG^{-1}'s. That is, when the original steady-state gain matrix is partitioned as follows:

$$G_1(0) = \left[\begin{array}{cc|c} 4 & 4 & 4 \\ \hline 3 & -3 & 5 \end{array}\right] \quad (47)$$

the resulting $BRG^{-1}_{l_{11}} = 1.0$ and $BRG^{-1}_{l_{22}} = 1.0$. Permuting the matrix to obtain the following partitioning also results in $BRG^{-1}_{l_{11}} = BRG^{-1}_{l_{22}} = 1.0$:

$$G_2(0) = \left[\begin{array}{c|cc} 4 & 4 & 4 \\ \hline 5 & 3 & -3 \end{array}\right] \quad (48)$$

Therefore, two desirable partitionings exist for the nonsquare system. The first has the pairings $(y_1; u_1, u_2)$ and $(y_2; u_3)$ while the second has the pairings (y_1, u_1) and $(y_2; u_2, u_3)$. The relative sensitivity asymptotes may be employed to eliminate the less desirable alternative. The asymptotes for the first structure are:

$$(\bar{S}_{12})_1 = \frac{4}{5}\left(\frac{10s+1}{20s+1}\right) \quad (49)$$

$$(\bar{S}_{21})_1 = 0 \quad (50)$$

and for the second structure are:

$$(\bar{S}_{12})_2 = 0 \quad (51)$$

$$(\bar{S}_{21})_2 = \frac{5}{4}\left(\frac{20s+1}{10s+1}\right) \quad (52)$$

Both structures have one asymptote that is zero at all frequencies, indicating that only one-way interactions should exist. Figure 3 compares the two nonzero asymptotes over a large range of frequencies. From this comparison it is obvious that the second partitioning is less desirable than the first because of the larger asymptote at all frequencies. Therefore, the recommended partitioning would be the first, as shown in Eq. 47, where y_1 is paired with u_1 and u_2 while y_2 is paired with u_3.

Controller design based on DBRG^{-1} *and relative sensitivities*

IMC controllers will be employed, with the H_i's chosen to make the controllers strictly proper. The controllers for the first partitioning will therefore be:

$$(G_{l_1})_1 = ([G_{11}]^{\dagger}H_1)_1 = \begin{bmatrix} \frac{20s+1}{8} \\ \frac{20s+1}{8} \end{bmatrix} \frac{1}{(\epsilon_1 s+1)^2} \quad (53)$$

$$(G_{l_2})_1 = ([G_{22}]^{\dagger}H_2)_1 = \left(\frac{10s+1}{5}\right)\frac{1}{(\epsilon_2 s+1)^2} \quad (54)$$

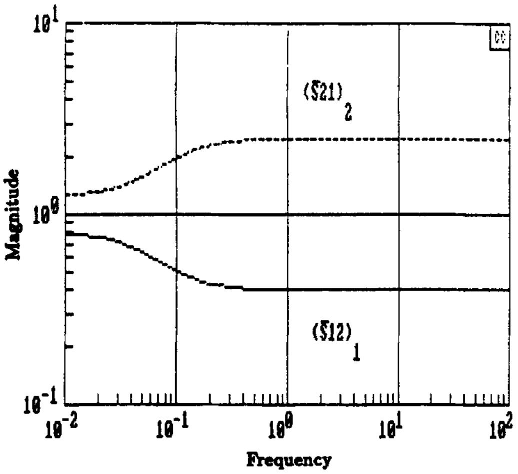

Figure 3. Relative sensitivity asymptotes for G_1 and G_2.

and similarly, for the second partitioning:

$$(G_{l_1})_2 = ([G_{11}]^{\dagger}H_1)_2 = \left(\frac{20s+1}{4}\right)\frac{1}{(\epsilon_1 s+1)^2} \tag{55}$$

$$(G_{l_2})_2 = ([G_{22}]^{\dagger}H_2)_2 = \begin{bmatrix} \dfrac{10s+1}{6} \\ \dfrac{10s+1}{-6} \end{bmatrix}\frac{1}{(\epsilon_2 s+1)^2} \tag{56}$$

Now the $DBRG^{-1}$'s for both partitionings may be calculated:

$$(DBRG_{l_{11}}^{-1})_1 = 1 - \left(\frac{4}{20s+1}\right)\left(\frac{10s+1}{5}\right)\frac{1}{(\epsilon_2 s+1)^2} \cdot \begin{bmatrix} \dfrac{3}{10s+1} & \dfrac{-3}{10s+1} \end{bmatrix}\begin{bmatrix} \dfrac{20s+1}{8} \\ \dfrac{20s+1}{8} \end{bmatrix} = 1 \tag{57}$$

$$(DBRG_{l_{22}}^{-1})_1 = 1 - \begin{bmatrix} \dfrac{3}{10s+1} & \dfrac{-3}{10s+1} \end{bmatrix}\begin{bmatrix} \dfrac{20s+1}{8} \\ \dfrac{20s+1}{8} \end{bmatrix} \cdot \frac{1}{(\epsilon_1 s+1)^2}\left(\frac{4}{20s+1}\right)\left(\frac{10s+1}{5}\right) = 1 \tag{58}$$

Therefore the local performances Y_{11} and Y_{22} can be expected to show no interactions for the first partitioning. Furthermore, these $DBRG^{-1}$ values simplify the stability considerations greatly for they imply that the system will be stable no matter what values are assigned to the ϵ's; see theorem 6. The second subsystem produces similar results, with $(DBRG_{l_{11}}^{-1})_2 = (DBRG_{l_{22}}^{-1})_2 = 1$.

The relative sensitivities should be examined next. For the first partitioning they will be

$$(S_{12})_1 = \frac{4}{5}\left(\frac{10s+1}{20s+1}\right)\left[1 - \frac{1}{(\epsilon_1 s+1)^2}\right] \tag{59}$$

$$(S_{21})_1 = 0 \tag{60}$$

and for the second partitioning will be

$$(S_{12})_2 = 0 \tag{61}$$

$$(S_{21})_2 = \frac{5}{4}\left(\frac{20s+1}{10s+1}\right)\left[1 - \frac{1}{(\epsilon_2 s+1)^2}\right] \tag{62}$$

Therefore one-way decoupling, $(Y_{21})_1 = (Y_{12})_2 = 0$, may be expected for both partitionings because of the zero values for the corresponding relative sensitivities. (Note that these values are consistent with the zero values of the asymptotes.) However, as Figure 4 illustrates, the nonzero relative sensitivity of the second structure is significantly higher in magnitude than that of the first structure for $\epsilon_1 = \epsilon_2 = 0.5$. As expected, each relative sensitivity approaches its asymptote at higher frequencies. The value of each ϵ was set at 0.5 to obtain good set-point responses. The higher relative sensitivity predicts that for such small ϵ's, larger interactions will be produced for the second structure than for the first. The simulations of Figures 5 and 6 confirm this. Therefore these simulations have shown that the relative sensitivity asymptotes correctly identified the superior nonsquare partitioning. The simulations also illustrate the one-way decoupling predicted by the relative sensitivities. Furthermore, note that $Y_{11} = Y_{22} = H_1 = H_2 = 1/(0.5s + 1)^2$ for both structures because the $DBRG^{-1}$ values were 1 and the ϵ's were 0.5. That is, the simulations demonstrate that having $DBRG^{-1}$'s equal to 1 means that each subsystem attains the *local* closed-loop performance it would have if it were isolated from the other subsystem.

The performance of the best nonsquare partitioning should also be compared to that of the best square partitioning. Comparing Figure 5 and Figure 7 demonstrates the superiority of the chosen nonsquare control system over the square system with

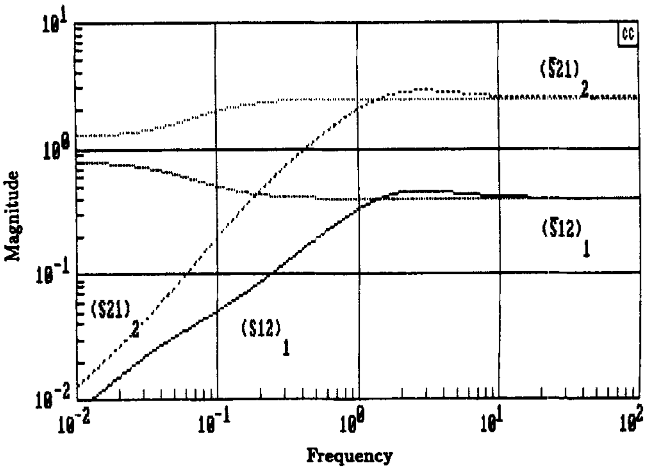

Figure 4. Relative sensitivities for G_1 and G_2.
$\epsilon_1 = \epsilon_2 = 0.5$

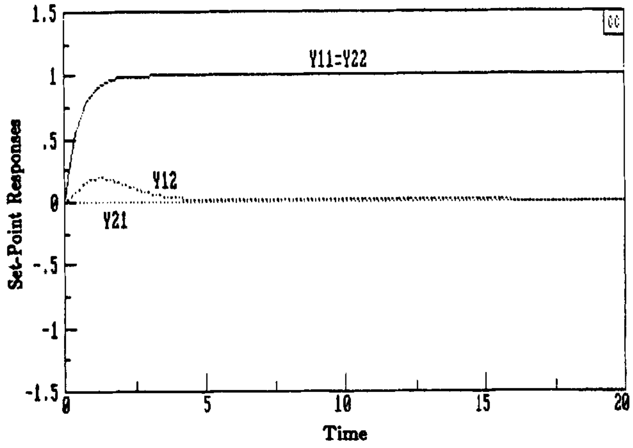

Figure 5. Set-point responses of nonsquare system G_1. $\epsilon_1 = \epsilon_2 = 0.5$

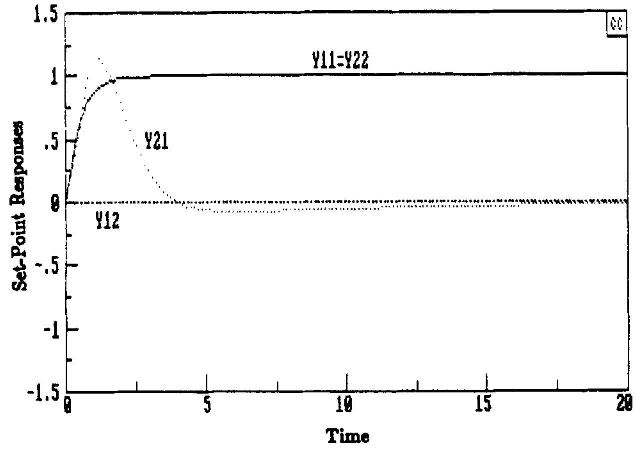

Figure 6. Set-point responses of nonsquare system G_2. $\epsilon_1 = \epsilon_2 = 0.5$

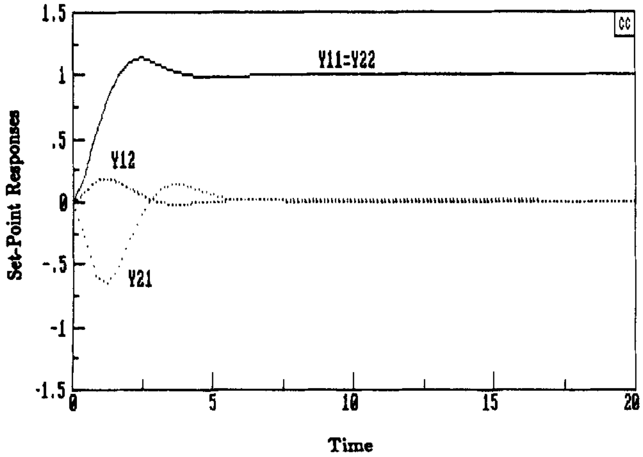

Figure 7. Set-point responses of the "best" square system. $\epsilon_1 = \epsilon_2 = 0.5$

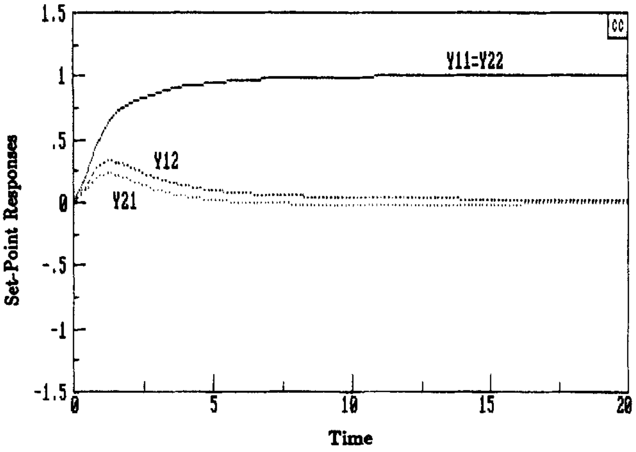

Figure 8. Set-point responses of nonsquare system G_3. $\epsilon_1 = \epsilon_2 = 0.5$

the pairings (y_1, u_2) and (y_2, u_3). A faster response with smaller interactions is achieved by the nonsquare system when the ϵ's are set at 0.5 for both systems. Furthermore, since the $DBRG^{-1}$'s for the nonsquare system equal 1 at all frequencies, the values of the ϵ's can be decreased further to improve performance without encountering stability problems (assuming there is no plant uncertainty). This is not the case for the square system where further decreases in the ϵ's to improve performance will eventually sacrifice stability. The performance of a nonsquare control system with nonidentity BRG^{-1}'s may be observed in Figure 8. This system employs the structure:

$$G_3(0) = \left[\begin{array}{cc|c} 4 & 4 & 4 \\ \hline -3 & 5 & 3 \end{array}\right] \tag{63}$$

where y_1 is paired with u_2 and u_3 while y_2 is paired with u_1. The block relative gains for this partitioning are $BRG^{-1}_{l_{11}} = BRG^{-1}_{l_{22}} = 2/3$. Consequently, poorer set-point responses will be expected from a control system employing this structure. Furthermore, the relative sensitivity asymptotes for this partitioning predict slightly larger interactions than those present in the best nonsquare structure. Figure 8 confirms these expectations for ϵ's of 0.5.

To summarize, this example has illustrated the predictive ability of the BRG^{-1} and relative sensitivity asymptotes as well as the possible inherent superiority of a nonsquare system over its component square systems. Furthermore the use of the nonsquare $DBRG^{-1}$ and relative sensitivities in controller design and analysis has been demonstrated.

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